TOWARDS MATRIX MODELS OF IIB SUPERSTRINGS

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INTRODUCTION

Recently there has been a lot of papers on matrix models and superstrings, induced by the work of Banks, Fischler, Shenker, and Susskind (hep-th/ 9706168). I refer to Makeenko's talk at this meeting for a general review of this subject.

Most of the work* reported in this talk has been done together with Fayyazuddin, Makeenko, Smith, and Zarembo[1]. As explained in Makeenko's (virtual) talk at this meeting, we started from the work by Ishibashi, Kawai, Kitazawa, and Tsuchiya[2], who proposed that type IIB superstrings in 10 dimensions are described by the reduced action,

$$S_{\mathrm{IKKT}} = \alpha \left(-\frac{1}{4} \mathrm{Tr}\ [A_\mu, A_\nu]^2 - \frac{1}{2} \mathrm{Tr}\ (\bar{\psi}\Gamma^\mu [A_\mu, \psi]) \right), \tag{1}$$

where A_μ and ψ_α are $n \times n$ matrices. A sum over n is implied, with weight $\exp(-\beta n)$. Later the sum over n has been replaced by a double scaling limit[3].

In our paper[1] we discussed various problems associated with eq. (1), and we proposed a different model with action

$$S_{\mathrm{NBI}} = -\frac{\alpha}{4} \mathrm{Tr} \left(Y^{-1} [A_\mu, A_\nu]^2 \right) + V(Y) - \frac{\alpha}{2} \mathrm{Tr}\ (\bar{\psi}\Gamma^\mu [A_\mu, \psi]), \tag{2}$$

where the potential is given by

$$V(Y) = \beta\ \mathrm{Tr} Y + \gamma\ \mathrm{Tr} \ln Y. \tag{3}$$

*The paper in reference 1 has been published in Nuclear Physics B. Unfortunately, the editors of that journal used an early draft of the manuscript, which contained several typhos. For this reason I cannot recommend the published version, but refer to the version in the Archives.

The partition function is thus given by

$$Z = \int dA_\mu d\psi dY \; e^{-S_{\rm NBI}}. \tag{4}$$

We select the constant γ in such a way that the result of the Y-integration is as close to the superstring as is possible. This turns out to mean

$$\gamma = n - \frac{1}{2}, \tag{5}$$

as we shall see later.

Physically the model $S_{\rm NBI}$ is motivated by a GUT scenario: Suppose one has a field theory valid down to the GUT scale. Then, in our model the group is SU(n), with n large. As we shall see, this type of GUT model then leads to superstrings if $n \to \infty$. For n finite, supersymmetry is broken, as is expected for energies below the GUT energy. Thus, superstrings can emerge from a GUT type of model. Of course, the model with action $S_{\rm NBI}$ is not a realistic GUT model.

Under the SUSY transformations

$$\delta\psi = \frac{i}{4}\{Y^{-1},[A_\mu,A_\nu]\}\Gamma^{\mu\nu}\epsilon, \quad \delta A_\mu = i\bar{\epsilon}\Gamma_\mu\psi, \tag{6}$$

the action transforms like

$$\delta S_{\rm NBI} \propto \epsilon^{\mu\alpha\beta\lambda_1\ldots\lambda_7}\; {\rm Tr}\left(\psi_m(\Gamma^0\Gamma^{11}\Gamma^{\lambda_1}\ldots\Gamma^{\lambda_7})_{mp}\epsilon_p\{[A_\alpha,A_\beta],[A_\mu,Y^{-1}]\}\right). \tag{7}$$

It can be shown that

$$\delta S_{\rm NBI} \to 0 \;\; \text{for} \;\; n \to \infty, \tag{8}$$

so the action is supersymmetric in the limit $n \to \infty$, but for finite n the symmetry is broken.

THE Y-INTEGRATION

The integration over Y can be done exactly. Consider

$$\mathcal{F}(z) = \int dY \;\exp\left(-\frac{\alpha}{4}{\rm Tr}(Y^{-1}z^2) - \beta\; {\rm Tr}Y - \gamma\; {\rm Tr}\ln Y\right), \quad z^2 \equiv -[A_\mu,A_\nu]^2. \tag{9}$$

The "angular" integration is of the Itzykson-Zuber type, so we get

$$\mathcal{F}(z) = {\rm const.}\prod_{i=1}^{i=n}\int dy_i \frac{\Delta^2(y)}{\Delta(1/y)\Delta(z^2)}\; e^{-\alpha\sum_i z_i^2/4y_i - \beta\sum_i y_i - \gamma\sum_i \ln y_i}. \tag{10}$$

Here the z_i's and y_i's are the eigenvalues, and

$$\Delta(x) = \prod_{i>j}(x_i - x_j) = \det_{ki} x_i^{k-1} \tag{11}$$

is the Vandermonde determinant. We only integrate over the positive eigenvalues of Y. Thus we get

$$\Delta(z^2)\mathcal{F}(z) = {\rm const.}\int_0^\infty \prod_i dy_i\; y_i^{n-1}\prod_{i>j}(y_i - y_j)\; e^{-\alpha\sum_i z_i^2/4y_i - \beta\sum_i y_i - \gamma\sum_i \ln y_i}. \tag{12}$$

This can be rewritten as a determinant

$$\Delta(z^2)\mathcal{F}(z) = \text{const.} \det_{ki} \int \frac{dy}{\sqrt{y}} y^{k-1} e^{-\alpha z_i^2/4y - \beta y}$$
$$= \text{const.} \det_{ki} \left[(-1)^{k-1} \frac{\partial^{k-1}}{\partial \beta^{k-1}} \left(\sqrt{\frac{\pi}{\beta}}\, e^{-\sqrt{\alpha\beta} z_i} \right) \right]. \tag{13}$$

Here z_i is by definition the positive square root of z_i^2. This determinant can be evaluated using basic properties of determinants, and the result is[1]

$$\Delta(z^2)\mathcal{F}(z) = \text{const.}\ \Delta(z)\ e^{-\sqrt{\alpha\beta}\ \sum_i z_i}. \tag{14}$$

This result is exact, and hence it is valid for any n.

The sum over eigenvalues in the exponent has the following interpretation,

$$\sum_i z_i = \sum_i \sqrt{z_i^2} = \frac{1}{4i\sqrt{\pi}} \int_{-\infty}^{(0+)} \frac{dt}{t^{3/2}} \sum_i e^{t z_i^2}, \tag{15}$$

where we used an integral representation of the square root. Thus,

$$\sum_i z_i = \frac{1}{4i\sqrt{\pi}} \int_{-\infty}^{(0+)} \frac{dt}{t^{3/2}}\ \text{Tr}\ \exp(-t[A_\mu, A_\nu]^2) = \text{Tr}\ \sqrt{-[A_\mu, A_\nu]^2}, \tag{16}$$

valid in Euclidean space.

The partition function therefore becomes

$$Z = \int dA_\mu d\psi dY\ \exp\left(\frac{\alpha}{4} \text{Tr}(Y^{-1}[A_\mu, A_\nu]^2) - \beta\ \text{Tr} Y - \gamma\ \text{Tr} \ln Y \right)$$
$$= \text{const.} \int \frac{dA_\mu d\psi}{\prod_{i>j}(z_i + z_j)} \exp\left(-\sqrt{\alpha\beta}\ \text{Tr}\sqrt{-[A_\mu, A_\nu]^2} - \frac{\alpha}{2} \text{Tr}\ (\bar{\psi}\Gamma^\mu [A_\mu, \psi]) \right). \tag{17}$$

This is the exact result of the Y-integration. In order to get the square root it is important to use the value of γ given in eq. (5).

This result can be expressed in an alternative form, at the cost of introducing an auxillary Hermitean field M. We use the identity

$$\frac{1}{\prod_{i<j}(z_i + z_j)} = \text{const.} \sqrt{\det z} \int dM e^{-\text{Tr}\ z M^2}, \tag{18}$$

to obtain

$$Z = \text{const.} \int dA_\mu d\psi dM \left(\det \sqrt{-[A_\mu, A_\nu]^2} \right)^{1/2}$$
$$\times \exp\left(-\text{Tr}\left((\sqrt{\alpha\beta} + M^2)\sqrt{-[A_\mu, A_\nu]^2} \right) - \frac{\alpha}{2} \text{Tr}\ (\bar{\psi}\Gamma^\mu [A_\mu, \psi]) \right). \tag{19}$$

The field M is essentially trivial, with a "classical equation of motion" $M = 0$.

The square root occuring in the result above is somewhat reminisent of the Nambu-Goto square root. If we could replace the commutator in the square root in eq. (17) by the corresponding Poisson bracket, we would have a partition function which is very similar to the one for the superstring.

This problem has been discussed by Hoppe[4], and in different settings by a number of other authors[5,6, 7]. It turns out that making some assumptions, one has the limit

$$\text{“ } [A, B] \to i\{A, B\}_{\mathrm{PB}} \text{ “}, \quad \text{for} \quad n \to \infty. \tag{20}$$

Here $\{,\}_{\mathrm{PB}}$ denotes the usual Poisson bracket.

We refer to the literature for a detailed discussion. Here we shall follow Bars[7], and consider a torus (although this restriction is probably not important[8]). The case of a sphere was discussed in ref. 5. A Hermitean matrix can be expanded in a Weyl basis,

$$(A_\mu)^i_j = C_1 \sum_{\mathbf{k}} a^{\mathbf{k}}_\mu (l_{\mathbf{k}})^i_j, \quad \text{with} \quad \mathbf{k} = (k_1, k_2). \tag{21}$$

The matrix $l_{\mathbf{k}}$ can be expressed in terms of the $n \times n$ (n =odd) Weyl matrices h and g, which satisfy

$$h^n = g^n = 1 \quad \text{and} \quad gh = \omega hg, \quad \text{with} \quad \omega = e^{4\pi i/n}. \tag{22}$$

The explicit form of these matrices are

$$\begin{aligned} h &= \mathrm{diag}(1, \omega, \omega^2, ..., \omega^{n-1}) \\ g^i_j &= \delta^i_{j+1}, \ i = 1, 2, ..., n-1, \ g^n_j = 0 \text{ except for } g^n_1 = 1. \end{aligned} \tag{23}$$

The SU(n)-generators $l_{\mathbf{k}}$ are then constructed as

$$l_{\mathbf{k}} = \frac{n}{4\pi} \omega^{k_1 k_2/2} h^{k_1} g^{k_2}, \tag{24}$$

since the powers of h and g are linearly independent for $k_1, k_2 = 1, 2, ..., n-1$, are unitary, close under multiplication, and are traceless. Using that

$$\mathrm{Tr}\ h^{k_1} g^{k_2} = n \delta_{k_1,0} \delta_{k_2,0}, \tag{25}$$

we easily see that

$$\mathrm{Tr}\ l_{\mathbf{k}} l_{\mathbf{p}} = \frac{n^3}{(4\pi)^2} \delta_{\mathbf{k}+\mathbf{p},\mathbf{0}}. \tag{26}$$

Thus the expansion coefficients in eq. (21) are given by

$$a^{-\mathbf{p}}_\mu = (a^{\mathbf{p}}_\mu)^* = \frac{(4\pi)^2}{n^3 C_1} \mathrm{Tr}\ (l_{\mathbf{p}} A_\mu). \tag{27}$$

Also, from the relation

$$g^{k_1} h^{k_2} = \omega^{k_1 k_2} h^{k_2} g^{k_1} \tag{28}$$

we get

$$[l_{\mathbf{p}}, l_{\mathbf{k}}] = i \frac{n}{2\pi} \ \sin\left(\frac{2\pi}{n} \mathbf{p} \times \mathbf{k}\right) l_{\mathbf{p}+\mathbf{k}}, \tag{29}$$

where

$$\mathbf{p}\times\mathbf{k}=p_1k_2-k_1p_2. \tag{30}$$

Using the expansion (21) we get

$$[A_\mu,A_\nu]_i^j=C_1^2\sum_{\mathbf{p},\mathbf{q}}a_\mu^{\mathbf{p}}a_\nu^{\mathbf{q}}\frac{n}{2\pi}\sin\left(\frac{2\pi}{n}\mathbf{p}\times\mathbf{q}\right)(l_{\mathbf{p}+\mathbf{q}})_i^j. \tag{31}$$

This can be compared with the similar expression for the string variables $X_\mu(\sigma,\tau)$, where we have the expansion

$$X_\mu(\sigma,\tau)=C_2\sum_{\mathbf{m}}a_\mu^{\mathbf{m}}\exp(i\sigma m_1+i\tau m_2), \tag{32}$$

leading to

$$\{X_\mu,X_\nu\}_{\rm PB}=C_2^2\sum_{\mathbf{p},\mathbf{q}}a_\mu^{\mathbf{p}}a_\nu^{\mathbf{q}}\;(\mathbf{p}\times\mathbf{q})\exp(i(\mathbf{p}+\mathbf{q})\sigma). \tag{33}$$

Now it is clear that *if*

$$\lim_{n\to\infty}\sum_{\rm modes}...=\sum_{\rm modes}\lim_{n\to\infty}..., \tag{34}$$

then we have by use of (26)

$$\mathrm{Tr}\;[A_\mu,A_\nu]^2\to\mathrm{const.}\int d\sigma d\tau\{X_\mu,X_\nu\}^2\quad\text{for}\quad n\to\infty. \tag{35}$$

It is obvious that the commutativity (34) is only valid if the infinite modes are unimportant. This is, however, not true e.g. for the bosonic string. If we fix the end points of this string at some distance, then there is a critical distance (essentially the inverse tachyon mass) at which the string oscillates so wildly that this behaviour can only be reproduced with an infinite number of modes. Below this distance the "string" becomes a branched polymer, and hence is no longer a string.

For superstrings this problem does not arize, and hence there is at least no obvious reason why the limits cannot be intechanged as in eq. (35). In the following we assume that eq. (35) is correct for type IIB superstrings.

Since we are interested in the square root of the squared commutator, the result (35) is not enough. Using eq. (26) and repeated applications of the formula

$$l_{\mathbf{m}}l_{\mathbf{r}}=(n/4\pi)\exp(2\pi i(\mathbf{m}\times\mathbf{r})/n))l_{\mathbf{m}+\mathbf{r}} \tag{36}$$

one can easily derive

$$\mathrm{Tr}\;l_{\mathbf{m_1}}l_{\mathbf{m_2}}...l_{\mathbf{m_s}}\to n(n/4\pi)^s\;\delta_{\mathbf{m_1}+\mathbf{m_2}+...+\mathbf{m_s},\mathbf{0}},\;\text{for}\;n\to\infty \tag{37}$$

to leading order in n. Using this in eq. (16) we obtain

$$\begin{aligned}\mathrm{Tr}\;\sqrt{-[A_\mu,A_\nu]^2}\;&=\;\frac{1}{4i\sqrt{\pi}}\int_{-\infty}^{(0+)}\frac{dt}{t^{3/2}}\mathrm{Tr}\;\exp(-t[A_\mu,A_\nu]^2)\\&\to\frac{1}{4i\sqrt{\pi}}\int_{-\infty}^{(0+)}\frac{dt}{t^{3/2}}\int d\sigma d\tau\exp(t\;\mathrm{const.}\{X_\mu,X_\nu\}_{\rm PB}^2)\\&=\mathrm{const.}\int d\sigma d\tau\sqrt{\{X_\mu,X_\nu\}_{\rm PB}^2}\;\;\text{for}\;n\to\infty.\end{aligned} \tag{38}$$

Thus we see that in the leading order the Nambu-Goto square root arises as the limit of the square root of the corresponding commutator. However, it should be remembered what was said before about strings with tachyons. They do not allow the interchange of limits as in (34), and hence the result (38) is not valid in that case[†].

TYPE IIB SUPERSTRING FROM THE NBI MATRIX MODEL

We can now summarize our results in the following rather long formula,

$$
\begin{aligned}
Z &= \int dA_\mu d\psi dY \;\exp\left(\frac{\alpha}{4}\mathrm{Tr}(Y^{-1}[A_\mu, A_\nu]^2) - \beta\;\mathrm{Tr}Y - (n-\frac{1}{2})\;\mathrm{Tr}\ln Y\right) \\
&= \text{const.}\int dA_\mu d\psi dM \left(\det\sqrt{-[A_\mu, A_\nu]^2}\right)^{1/2} \\
&\quad \times \exp\left(-\mathrm{Tr}\left((\sqrt{\alpha\beta}+M^2)\sqrt{-[A_\mu, A_\nu]^2}\right) - \frac{\alpha}{2}\mathrm{Tr}\;(\bar{\psi}\Gamma^\mu[A_\mu, \psi])\right) \\
&\to \text{const.}\int dX_\mu d\psi dM \left(\det\sqrt{\{X_\mu, X_\nu\}^2}\right)^{1/2} \\
&\quad \times \exp\left(-\int d\sigma d\tau\left((\sqrt{\alpha\beta}+M^2)\sqrt{\{X_\mu, X_\nu\}^2} - \frac{i\alpha}{2}(\bar{\psi}\Gamma^\mu\{X_\mu, \psi\})\right)\right),
\end{aligned}
\tag{39}
$$

where the last expression is valid for $n \to \infty$. In this expression the fields ψ and M have expansions similar to eqs. (21) and (32), and some normalization constants have been absorbed in α and β in the last formula above.

The functional integration over M in eq. (39) is just Gaussian and can of course easily be performed,

$$
\begin{aligned}
z \to \text{const.}&\int dX_\mu d\psi \left(\frac{\det\sqrt{\{X_\mu, X_\nu\}^2}}{\mathcal{D}et\sqrt{\{X_\mu, X_\nu\}^2}}\right)^{1/2} \\
&\times \exp\left(-\int d\sigma d\tau\left(\sqrt{\alpha\beta}\sqrt{\{X_\mu, X_\nu\}^2} - \frac{i\alpha}{2}(\bar{\psi}\Gamma^\mu\{X_\mu, \psi\})\right)\right).
\end{aligned}
\tag{40}
$$

The two determinants in this expression arises from different types of Gaussian integrations, the "det" beeing defined through eq. (18) and the subsequent limit $n \to \infty$, whereas the "$\mathcal{D}et$" determinant comes from the continuum integral over M. Naively one would tend to identify these two determinants, so that the fraction containing them is just one,

$$
\frac{\det\sqrt{\{X_\mu, X_\nu\}^2}}{\mathcal{D}et\sqrt{\{X_\mu, X_\nu\}^2}} \to 1.
\tag{41}
$$

If so, the NBI action gives exactly the Nambu-Goto version of the Green-Schwarz type IIB superstring.

However, Zarembo[9] has pointed out to me that the situation can be more complicated. For example, from eq. (18) one sees that the "det" determinant (= $\det z$ for

[†]For such strings where the end points are actually separated by a *large* distance, the interchange of limits in (34) is probably allowed. Thus at large distances the string picture is most likely right for the NBI matrix model even without supersymmetry. At shorter distances near the critical one, this picture breaks down, and the sine function in (31) cannot be approximated by its first term in a power series expansion.

$n \to \infty$) is subdominant relative to the factor $\prod(z_i + z_k)$ occurring in eq. (18), and hence can be ignored in the limit $n \to \infty$. In this case, one has instead of (41)

$$\frac{\det \sqrt{\{X_\mu, X_\nu\}^2}}{\mathcal{D}et\sqrt{\{X_\mu, X_\nu\}^2}} \to \frac{1}{\mathcal{D}et\sqrt{\{X_\mu, X_\nu\}^2}}. \tag{42}$$

If so, the $\mathcal{D}et$ determinant survives in the measure. However, even if this is so, this factor is rather harmless: correlation functions are invariant, since the measure is multiplied by a constant factor under reparametrizations. This factor does not depend on the fields and cancels in the correlation functions[9].

Perhaps the right answer depends on how exactly the continuum limit is constructed, because in order to interpret $\mathcal{D}et$ a regulator is needed.

It should also be mentioned that Chekhov and Zarembo[10] have discussed models somewhat different from the NBI model, and have also discussed the measure in more details.

A SADDLE POINT AND THE VIRTUAL EULER NUMBER

We shall now study the saddle point of the NBI action. By variation of the A_μ−fields we obtain the classical equation of motion

$$[A_\mu, \{Y^{-1}, [A_\mu, A_\nu]\}] = 0. \tag{43}$$

This equation was studied by Kristjansen and me[11]. The solution is

$$[A_\mu, A_\nu]^i_j = i m_{\mu\nu}\, Y^i_j, \tag{44}$$

where $m_{\mu\nu}$ is a matrix with repect to the space indices. In the saddle point the action has the value

$$S^{\text{saddle}}_{\text{NBI}} = (\beta + m^2_{\mu\nu}\alpha/4)\ \text{Tr}Y + (n - 1/2)\ \text{Tr}\ln Y. \tag{45}$$

In order to have a non-trivial $n \to \infty$ limit, it is necessary that α and β are of order n. It should be stressed that this does *not* imply the usual classical limit in string theory, as explained in details in ref. 11.

In addition to the terms exhibited above, there are of course subdominant terms arising from the expansion of A_μ around the classical solution. These terms are ignored in the following. Therefore, at the A_μ-saddle point we have the integral[11] (α/n and β/n are both of order one)

$$Z^{\text{saddle}} = \int dY \exp[-n \left\{(\beta + m^2_{\mu\nu}\alpha/4)/n\ \text{Tr}Y + t\ \text{Tr}\ln Y\right\}],\ \text{with}\ \ t = 1 - 1/2n. \tag{46}$$

This functional integral is of the Penner type[12]. For the value of the parameter t needed in the saddle point, the Y-integral actually diverges. However, by analytic continuation á la the gamma function for negative argument one can start by defining the integral for negative t, and then ultimately continue back to positive t. It the turns out that $t = 1$ is a critical point, and in the vicinity of this point one can define a double scaling limit with the “cosmological constant”

$$\mu = (1 - t)n = \text{fixed}. \tag{47}$$

We see that with the value of t given by eq. (46), μ =1/2=fixed quite automatically! Thus, we do not need to make any special assumptions in order to have this double scaling limit in the NBI model.

What is the meaning of the Penner model in the double scaling limit? An asymptotic expansion in μ can be made. Consider the "free energy" F, $Z^{\rm saddle} \equiv e^F$, then

$$F(\mu) = F_0(\mu) + F_1(\mu) + \sum_{g=2}^{\infty} \chi_g \mu^{2-2g}, \quad \mu = 1/2. \tag{48}$$

Here χ_g is the "virtual Euler number" for moduli space of Riemann surfaces with genus g, which is well known to be relevant for strings. One has[12]

$$\chi_g = \frac{B_{2g}}{2g(2g-2)}, \tag{49}$$

where B_{2g} are the Bernoulli numbers. These have positive sign, and blow up factorially, so the sum defining F is not Borel summable. This is also well known to be the case for genus expansion of string theories.

The physical interpretation of this result is that the field Y captures the Euler characteristic of moduli space of Riemann surfaces. Therefore it is quite likely that the NBI model encodes non-perturbative information on Riemann surfaces generated by moduli space. It should be remembered that in string theories one usually sum the functional integral over g, however here this seems to be already included. It must be admitted that the virtual Euler number represents very global properties of moduli space, and certainly more details are needed before one can claim a good understanding of the non-perturbative nature of this model.

Recently Soloviev[13] has commented on "a curious relation" between Siegel's model[14] of random lattice strings and the above saddle point approximation to the NBI model. This comes about if one starts from Siegel's T-self-dual matrix model

$$S = \mathrm{Tr}\left(\frac{1}{2}\Phi^2 + n\ \ln(1 - g\Phi)\right), \tag{50}$$

where Φ is a Hermitean $n \times n$ matrix and g is a constant. It was then pointed out by Soloviev[13] that if one makes the substitution

$$gY = 1 - g\Phi, \tag{51}$$

and perform the limit $n \to \infty,\ g \to 0,\ gn = $fixed, then one obtains

$$S \to n\ \mathrm{Tr}\ (\mathrm{const.} Y + n \ln Y) + \text{irrelevant const.}. \tag{52}$$

This is, however, precisely the saddle point expression (46) for the NBI model. This sadle point is therefore a weak string coupling limit of the Siegel matrix model[13]. For arbitrary coupling there is, however, an additional Y^2-term in the Siegel action, and hence it was suggested that perhaps the potential (3) should have an additional TrY^2 term[13]. Of course, a similar statement can be made about the NBI model, where there are various corrections to the saddle point expansion.

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